

New Non-Abelian Zeta Functions for Curves over Finite Fields

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In this paper, we introduce and study two new types of non-abelian zeta functions for curves over finite fields, which are defined by using (moduli spaces of) semi-stable vector bundles and non-stable bundles. A Riemann-Weil type hypothesis is formulated for zeta functions associated to semi-stable bundles, which we think is more canonical than the other one. All this is motivated by (and hence explains in a certain sense) our work on non-abelian zeta functions for number fields.

§1. Restricted Non-Abelian Zeta Functions

Let C be a regular, irreducible, projective curve of genus g defined over an algebraically closed field, and L a line bundle over C . Then, by a result of Mumford, we know that moduli space $\mathcal{M}_r(L)$ of semi-stable vector bundles of rank r over C with L as determinants is projective ([Mu]). And, if C and L are defined over the finite field \mathbf{F}_q with q elements, then by taking a finite base field extension if necessary, we may assume that $\mathcal{M}_r(L)$ is also defined over \mathbf{F}_q . Moreover, by a result of Harder and Narasimhan [HN], the \mathbf{F}_q -rational points of $\mathcal{M}_r(L)$ are exactly rank r semi-stable vector bundles of C defined over \mathbf{F}_q with L as determinants.

Clearly, the canonical line bundle K_C of C is defined over \mathbf{F}_q as well. Thus, for all $n \in \mathbf{Z}$, we obtain the following natural isomorphisms defined over \mathbf{F}_q too:

$$\begin{array}{ccc} \mathcal{M}_r(L) & \rightarrow & \mathcal{M}_r(L \otimes K_C^{\otimes nr}); \\ E & \mapsto & E \otimes K_C^{\otimes n}; \end{array} \quad \begin{array}{ccc} \mathcal{M}_r(L) & \rightarrow & \mathcal{M}_r(L^{\otimes -1} \otimes K_C^{\otimes nr}) \\ E & \mapsto & E^\vee \otimes K_C^{\otimes n}, \end{array}$$

where E^\vee denotes the dual of E .

Define the so-called Harder-Narasimhan number $M_{C,r,L}$ by setting

$$M_{C,r,L} := \sum_{E \in \mathcal{M}_r(L)(\mathbf{F}_q)} \frac{1}{\#\text{Aut}(E)}.$$

Here $\text{Aut}(E)$ denotes the automorphism group of E . (See e.g., [HN], [Se].) Easily, we have, for all $n \in \mathbf{Z}$,

$$M_{C,r,L} = M_{C,r,L \otimes K_C^{\otimes nr}} = M_{C,r,L^{\otimes -1} \otimes K_C^{\otimes nr}}.$$

The Harder-Narasimhan numbers may be calculated by counting contributions of non-stable bundles, with the help of a famous result of Siegel on quadratic forms and Harder-Narasimhan's filtration for vector bundles. (For details, see e.g., [HN], [DR] and [AB].) But in this section, we relate them with a new type of (restricted) non-abelian zeta functions.

Denote by h^i the dimension of cohomology groups H^i , $i = 0, 1$, by $\chi(C, E) := h^0(C, E) - h^1(C, E)$ the Euler-Poincaré characteristic, and by $d(E)$ the degree of E . Set

$$\mathcal{M}_{r,L}(\mathbf{F}_q) := \cup_{n \in \mathbf{Z}} \left(\mathcal{M}_r(L \otimes K_C^{\otimes nr})(\mathbf{F}_q) \cup \mathcal{M}_r(L^{\otimes -1} \otimes K_C^{\otimes nr})(\mathbf{F}_q) \right).$$

Definition. Let C be a regular, geometrically irreducible, projective curve defined over the finite field \mathbf{F}_q with q elements. With respect to any fixed positive integer $r \in \mathbf{Z}_{>0}$, and any line bundle L of C defined over \mathbf{F}_q , define a weight r and level L restricted non-abelian zeta function $\xi_{r,L}(s)$ of C by

$$\xi_{r,L}(s) := \sum_{E \in \mathcal{M}_{r,L}(\mathbf{F}_q)} \frac{q^{h^0(C,E)} - 1}{\#\text{Aut}(E)} \cdot \left(q^{\chi(C,E)} \right)^{-s}, \quad \text{Re}(s) > 1.$$

Clearly, $\xi_{r,L}(s) = \xi_{r,L \otimes \pm 1 \otimes K_C^{\otimes nr}}(s)$. Hence, from now on, we assume $0 \leq d(L) \leq r(g-1)$.

Theorem 1. *For $\operatorname{Re}(s) > 1$, we have*

$$\begin{aligned} \xi_{r,L}(s) = & \frac{1}{2} \sum_{E \in \mathcal{M}_{r,L}(\mathbf{F}_q); 0 \leq d(E) \leq r(2g-2)} \frac{q^{h^0(C,E)}}{\#\operatorname{Aut}(E)} \cdot \left((q^{\chi(C,E)})^{-s} + (q^{\chi(C,E)})^{s-1} \right) \\ & + \left[(q^{d(L)-r(g-1)})^{1-s} \cdot \frac{1}{q^{r(2g-2)(s-1)} - 1} + (q^{d(L)-r(g-1)})^s \frac{1}{q^{-sr(2g-2)} - 1} \right. \\ & \left. + (q^{d(L)-r(g-1)})^{s-1} \cdot \frac{1}{q^{r(2g-2)(s-1)} - 1} + (q^{d(L)-r(g-1)})^{-s} \frac{1}{q^{-sr(2g-2)} - 1} \right] \cdot M_{C,r,L}. \end{aligned} \quad (*)$$

Hence, in particular, we have

- (a) $\xi_{r,L}(s)$ can be meromorphically extended to the whole complex $t = q^{-s}$ -plane;
- (b) the extension, denoted also by $\xi_{r,L}(s)$, has simple poles at $s = 0$ and $s = 1$ with the Harder-Narasimhan number $M_{C,r,L}$ as residues;
- (c) $\xi_{r,L}(s)$ satisfies the functional equation $\xi_{r,L}(s) = \xi_{r,L}(1-s)$.

Proof of the Theorem. It suffices to prove (*). For this, we proceed as follows. Recall that for E semi-stable, if $d(E) > r(2g-2)$, then $h^1(C, E) = 0$; while if $d(E) < 0$, then $h^0(C, E) = 0$. Thus,

$$\begin{aligned} \xi_{r,L}(s) &= \sum_{E \in \mathcal{M}_{r,L}(\mathbf{F}_q); d(E) \geq 0} \frac{q^{h^0(C,E)} - 1}{\#\operatorname{Aut}(E)} \cdot (q^{\chi(C,E)})^{-s} \\ &= \sum_{E \in \mathcal{M}_{r,L}(\mathbf{F}_q); d(E) \geq 0} \frac{q^{h^0(C,E)}}{\#\operatorname{Aut}(E)} \cdot (q^{\chi(C,E)})^{-s} - \sum_{E \in \mathcal{M}_{r,L}(\mathbf{F}_q); d(E) \geq 0} \frac{1}{\#\operatorname{Aut}(E)} \cdot (q^{\chi(C,E)})^{-s} \\ &= \sum_{E \in \mathcal{M}_{r,L}(\mathbf{F}_q); 0 \leq d(E) \leq r(2g-2)} \frac{q^{h^0(C,E)}}{\#\operatorname{Aut}(E)} \cdot (q^{\chi(C,E)})^{-s} \\ &\quad + \sum_{E \in \mathcal{M}_{r,L}(\mathbf{F}_q); d(E) > r(2g-2)} \frac{q^{h^0(C,E)}}{\#\operatorname{Aut}(E)} \cdot (q^{\chi(C,E)})^{-s} - \sum_{E \in \mathcal{M}_{r,L}(\mathbf{F}_q); d(E) \geq 0} \frac{1}{\#\operatorname{Aut}(E)} \cdot (q^{\chi(C,E)})^{-s}. \end{aligned}$$

So $\xi_{r,L}(s) = S_{r,L}^{0 \leq d(E) \leq r(2g-2)}(s) + T_{r,L}(s)$ if we set

$$S_{r,L}^{0 \leq d(E) \leq r(2g-2)}(s) := \sum_{E \in \mathcal{M}_{r,L}(\mathbf{F}_q); 0 \leq d(E) \leq r(2g-2)} \frac{q^{h^0(C,E)}}{\#\operatorname{Aut}(E)} \cdot (q^{\chi(C,E)})^{-s},$$

and

$$T_{r,L}(s) := \sum_{E \in \mathcal{M}_{r,L}(\mathbf{F}_q); d(E) > r(2g-2)} \frac{q^{h^0(C,E)}}{\#\operatorname{Aut}(E)} \cdot (q^{\chi(C,E)})^{-s} - \sum_{E \in \mathcal{M}_{r,L}(\mathbf{F}_q); d(E) \geq 0} \frac{1}{\#\operatorname{Aut}(E)} \cdot (q^{\chi(C,E)})^{-s}.$$

Lemma. *The function $t^{r(g-1)} \cdot S_{r,L}^{0 \leq d(E) \leq r(2g-2)}(s)$ is holomorphic in $t = q^{-s}$. Moreover*

$$S_{r,L}^{0 \leq d(E) \leq r(2g-2)}(s) = \frac{1}{2} \sum_{E \in \mathcal{M}_{r,L}(\mathbf{F}_q); 0 \leq d(E) \leq r(2g-2)} \frac{q^{h^0(C,E)}}{\#\operatorname{Aut}(E)} \cdot \left((q^{\chi(C,E)})^{-s} + (q^{\chi(C,E)})^{s-1} \right)$$

and hence satisfies the functional equation

$$S_{r,L}^{0 \leq d(E) \leq r(2g-2)}(s) = S_{r,L}^{0 \leq d(E) \leq r(2g-2)}(1-s).$$

Proof of the lemma. Holomorphicity is clear, as only finitely many terms are involved. The functional equation comes from the Riemann-Roch theorem. Indeed,

$$\begin{aligned}
S_{r,L}^{0 \leq d(E) \leq r(2g-2)}(s) &= \frac{1}{2} \left(\sum_{E \in \mathcal{M}_{r,L}(\mathbf{F}_q); 0 \leq d(E) \leq r(2g-2)} \frac{q^{h^0(C,E)}}{\#\text{Aut}(E)} \cdot \left(q^{\chi(C,E)} \right)^{-s} \right. \\
&\quad \left. + \sum_{E \in \mathcal{M}_{r,L}(\mathbf{F}_q); 0 \leq d(E) \leq r(2g-2)} \frac{q^{h^0(C,E)}}{\#\text{Aut}(E)} \cdot \left(q^{\chi(C,E)} \right)^{-s} \right) \\
&= \frac{1}{2} \left(\sum_{E \in \mathcal{M}_{r,L}(\mathbf{F}_q); 0 \leq d(E) \leq r(2g-2)} \frac{q^{h^0(C,E)}}{\#\text{Aut}(E)} \cdot \left(q^{\chi(E)} \right)^{-s} \right. \\
&\quad \left. + \sum_{E^\vee \otimes K_C \in \mathcal{M}_{r,L}(\mathbf{F}_q); 0 \leq d(E^\vee \otimes K_C) \leq r(2g-2)} \frac{q^{h^0(C,E^\vee \otimes K_C)}}{\#\text{Aut}(E^\vee \otimes K_C)} \cdot \left(q^{\chi(C,E^\vee \otimes K_C)} \right)^{-s} \right) \\
&= \frac{1}{2} \left(\sum_{E \in \mathcal{M}_{r,L}(\mathbf{F}_q); 0 \leq d(E) \leq r(2g-2)} \frac{q^{h^0(C,E)}}{\#\text{Aut}(E)} \cdot \left(q^{\chi(C,E)} \right)^{-s} \right. \\
&\quad \left. + \sum_{E \in \mathcal{M}_{r,L}(\mathbf{F}_q); 0 \leq d(E) \leq r(2g-2)} \frac{q^{h^0(C,E^\vee \otimes K_C)}}{\#\text{Aut}(E)} \cdot \left(q^{\chi(C,E^\vee \otimes K_C)} \right)^{-s} \right) \\
&= \frac{1}{2} \left(\sum_{E \in \mathcal{M}_{r,L}(\mathbf{F}_q); 0 \leq d(E) \leq r(2g-2)} \frac{q^{h^0(C,E)}}{\#\text{Aut}(E)} \cdot \left(q^{\chi(C,E)} \right)^{-s} \right. \\
&\quad \left. + \sum_{E \in \mathcal{M}_{r,L}(\mathbf{F}_q); 0 \leq d(E) \leq r(2g-2)} \frac{q^{h^1(C,E^\vee \otimes K_C) + d(E^\vee \otimes K_C) - r(g-1)}}{\#\text{Aut}(E)} \cdot \left(q^{\chi(C,E^\vee \otimes K_C)} \right)^{-s} \right) \\
&= \frac{1}{2} \left(\sum_{E \in \mathcal{M}_{r,L}(\mathbf{F}_q); 0 \leq d(E) \leq r(2g-2)} \frac{q^{h^0(C,E)}}{\#\text{Aut}(E)} \cdot \left(q^{\chi(C,E)} \right)^{-s} \right. \\
&\quad \left. + \sum_{E \in \mathcal{M}_{r,L}(\mathbf{F}_q); 0 \leq d(E) \leq r(2g-2)} \frac{q^{h^0(C,E) - d(E) + r(g-1)}}{\#\text{Aut}(E)} \cdot \left(q^{-d(E) + r(g-1)} \right)^{-s} \right) \\
&= \frac{1}{2} \sum_{E \in \mathcal{M}_{r,L}(\mathbf{F}_q); 0 \leq d(E) \leq r(2g-2)} \frac{q^{h^0(C,E)}}{\#\text{Aut}(E)} \cdot \left(\left(q^{\chi(C,E)} \right)^{-s} + \left(q^{\chi(C,E)} \right)^{s-1} \right).
\end{aligned}$$

This completes the proof of the lemma.

As for $T_{r,L}(s)$, clearly, by the vanishing recalled at the beginning and the Riemann-Roch, we get

$$\begin{aligned}
T_{r,L}(s) &= \sum_{E \in \mathcal{M}_{r,L}(\mathbf{F}_q); d(E) > r(2g-2)} \frac{q^{d(E) - r(g-1)}}{\#\text{Aut}(E)} \cdot \left(q^{\chi(C,E)} \right)^{-s} - \sum_{E \in \mathcal{M}_{r,L}(\mathbf{F}_q); d(E) \geq 0} \frac{1}{\#\text{Aut}(E)} \cdot \left(q^{\chi(C,E)} \right)^{-s} \\
&= \sum_{E \in \mathcal{M}_{r,L}(\mathbf{F}_q); d(E) > r(2g-2)} \frac{1}{\#\text{Aut}(E)} \cdot \left(q^{\chi(C,E)} \right)^{1-s} - \sum_{E \in \mathcal{M}_{r,L}(\mathbf{F}_q); d(E) \geq 0} \frac{1}{\#\text{Aut}(E)} \cdot \left(q^{\chi(C,E)} \right)^{-s} \\
&= \left(\sum_{E \in \mathcal{M}_r(L \otimes K_C^{\otimes rn})(\mathbf{F}_q); d(E) > r(2g-2)} + \sum_{E \in \mathcal{M}_r(L^{\otimes -1} \otimes K_C^{\otimes rn})(\mathbf{F}_q); d(E) > r(2g-2)} \right) \cdot \frac{1}{\#\text{Aut}(E)} \cdot \left(q^{\chi(C,E)} \right)^{1-s} \\
&\quad - \left(\sum_{E \in \mathcal{M}_r(L \otimes K_C^{\otimes rn})(\mathbf{F}_q); d(E) \geq 0} + \sum_{E \in \mathcal{M}_r(L^{\otimes -1} \otimes K_C^{\otimes rn})(\mathbf{F}_q); d(E) \geq 0} \right) \cdot \frac{1}{\#\text{Aut}(E)} \cdot \left(q^{\chi(C,E)} \right)^{-s}
\end{aligned}$$

$$\begin{aligned}
&= \left(\sum_{E \in \mathcal{M}_r(L \otimes K_C^{\otimes rn})(\mathbf{F}_q); d(E) > r(2g-2)} \frac{1}{\#\text{Aut}(E)} \cdot \left(q^{\chi(C,E)} \right)^{1-s} \right. \\
&\quad - \sum_{E \in \mathcal{M}_r(L^{\otimes -1} \otimes K_C^{\otimes rn})(\mathbf{F}_q); d(E) \geq 0} \frac{1}{\#\text{Aut}(E)} \cdot \left(q^{\chi(C,E)} \right)^{-s} \\
&\quad + \left(\sum_{E \in \mathcal{M}_r(L^{\otimes -1} \otimes K_C^{\otimes rn})(\mathbf{F}_q); d(E) > r(2g-2)} \frac{1}{\#\text{Aut}(E)} \cdot \left(q^{\chi(C,E)} \right)^{1-s} \right. \\
&\quad \left. - \sum_{E \in \mathcal{M}_r(L \otimes K_C^{\otimes rn})(\mathbf{F}_q); d(E) \geq 0} \frac{1}{\#\text{Aut}(E)} \cdot \left(q^{\chi(C,E)} \right)^{-s} \right) \\
&= \left[\left(\sum_{n; d(E)=d(L)+nr(2g-2) > r(2g-2)} \left(q^{\chi(C,E)} \right)^{1-s} - \sum_{n; d(E)=-d(L)+nr(2g-2) \geq 0} \left(q^{\chi(C,E)} \right)^{-s} \right) \right. \\
&\quad \left. + \left(\sum_{n; d(E)=-d(L)+nr(2g-2) > r(2g-2)} \left(q^{\chi(C,E)} \right)^{1-s} - \sum_{n; d(E)=d(L)+nr(2g-2) \geq 0} \left(q^{\chi(C,E)} \right)^{-s} \right) \right] \cdot M_{C,r,L},
\end{aligned}$$

in which we omit stating that E 's are semi-stable with determinants $L^{\otimes \pm} \otimes K_C^{\otimes nr}$. Therefore,

$$\begin{aligned}
T_{r,L}(s) &= \left[\left(\sum_{n=1}^{\infty} \left(q^{d(L)+nr(2g-2)-r(g-1)} \right)^{1-s} - \sum_{n=1}^{\infty} \left(q^{-d(L)+nr(2g-2)-r(g-1)} \right)^{-s} \right) \right. \\
&\quad \left. + \left(\sum_{n=2}^{\infty} \left(q^{-d(L)+nr(2g-2)-r(g-1)} \right)^{1-s} - \sum_{n=0}^{\infty} \left(q^{d(L)+nr(2g-2)-r(g-1)} \right)^{-s} \right) \right] \cdot M_{C,r,L} \\
&= \left[\left(\sum_{n=1}^{\infty} \left(q^{\chi(E_0)+nr(2g-2)} \right)^{1-s} - \sum_{n=1}^{\infty} \left(q^{-\chi(C,E_0)+nr(2g-2)} \right)^{-s} \right) \right. \\
&\quad \left. + \left(\sum_{n=2}^{\infty} \left(q^{-\chi(C,E_0)+(n-1)r(2g-2)} \right)^{1-s} - \sum_{n=0}^{\infty} \left(q^{\chi(C,E_0)+nr(2g-2)} \right)^{-s} \right) \right] \cdot M_{C,r,L},
\end{aligned}$$

for any $E_0 \in \mathcal{M}_r(L)(\mathbf{F}_q)$. Thus, if $q^{r(2g-2)(1-s)} < 1$ and $q^{-sr(2g-2)} < 1$, by $0 \leq d(L) \leq r(g-1)$,

$$\begin{aligned}
T_{r,L}(s) &= \left[\left(\sum_{n=1}^{\infty} \left(q^{\chi(C,E_0)+nr(2g-2)} \right)^{1-s} - \sum_{n=1}^{\infty} \left(q^{-\chi(C,E_0)+(n-1)r(2g-2)} \right)^{-s} \right) \right. \\
&\quad \left. + \left(\sum_{n=2}^{\infty} \left(q^{-\chi(C,E_0)+(n-1)r(2g-2)} \right)^{1-s} - \sum_{n=0}^{\infty} \left(q^{\chi(C,E_0)+nr(2g-2)} \right)^{-s} \right) \right] \cdot M_{C,r,L} \\
&= \left[\left(q^{\chi(C,E_0)} \right)^{1-s} \sum_{n=1}^{\infty} \left(q^{r(2g-2)(1-s)} \right)^n - \left(q^{\chi(C,E_0)} \right)^s \sum_{n=0}^{\infty} \left(q^{-sr(2g-2)} \right)^n \right. \\
&\quad \left. + \left(q^{\chi(C,E_0)} \right)^{s-1} \sum_{n=1}^{\infty} \left(q^{r(2g-2)(1-s)} \right)^n - \left(q^{\chi(C,E_0)} \right)^{-s} \sum_{n=0}^{\infty} \left(q^{-sr(2g-2)} \right)^n \right] \cdot M_{C,r,L} \\
&= \left[\left(q^{\chi(C,E_0)} \right)^{1-s} \cdot \frac{q^{r(2g-2)(1-s)}}{1 - q^{r(2g-2)(1-s)}} + \left(q^{\chi(C,E_0)} \right)^s \frac{1}{q^{-sr(2g-2)} - 1} \right. \\
&\quad \left. + \left(q^{\chi(C,E_0)} \right)^{s-1} \cdot \frac{q^{r(2g-2)(1-s)}}{1 - q^{r(2g-2)(1-s)}} + \left(q^{\chi(C,E_0)} \right)^{-s} \frac{1}{q^{-sr(2g-2)} - 1} \right] \cdot M_{C,r,L} \\
&= \left[\left(q^{\chi(C,E_0)} \right)^{1-s} \cdot \frac{1}{q^{r(2g-2)(s-1)} - 1} + \left(q^{\chi(C,E_0)} \right)^s \frac{1}{q^{-sr(2g-2)} - 1} \right. \\
&\quad \left. + \left(q^{\chi(C,E_0)} \right)^{s-1} \cdot \frac{1}{q^{r(2g-2)(s-1)} - 1} + \left(q^{\chi(C,E_0)} \right)^{-s} \frac{1}{q^{-sr(2g-2)} - 1} \right] \cdot M_{C,r,L}.
\end{aligned}$$

This then proves the existence of meromorphic extension, the statement for simple poles and their residues, and the functional equation for $T_{r,L}(s)$. Thus by the Lemma, we complete the proof of the Theorem.

§2. New Non-Abelian Zeta Functions for Curves over Finite Fields

From definition, clearly,

$$\xi_{1,L}(s) = q^{(g-1)s} \sum_{E=L \otimes^\pm \otimes K_C^{\otimes n}; n \in \mathbf{Z}} \frac{q^{h^0(C,E)} - 1}{q - 1} \cdot (q^{d(E)})^{-s}.$$

So, by taking a finite sum over suitable L , we could arrive at

$$q^{(g-1)s} \sum_{M \in \text{Pic}(C)(\mathbf{F}_q)} \frac{q^{h^0(C,E)} - 1}{q - 1} \cdot (q^{d(E)})^{-s},$$

which is nothing but

$$q^{(g-1)s} \cdot \sum_{D \geq 0} N(D)^{-s},$$

i.e., the *standard abelian zeta function* $\zeta_C(s)$, or Artin zeta function $([A])$, for C times with $q^{(g-1)s}$. This then suggests the the following discussion.

Denote by $\mathcal{M}_{r,d}(C)$ the moduli space of degree d semi-stable vector bundles of rank r on C . Fixed a degree 1 line bundle A on C which is \mathbf{F}_q -rational. (One may give a proof of this fact by using properties of Artin zeta function $\zeta_C(s)$.) Clearly, we then have the following isomorphisms defined over \mathbf{F}_q :

$$\begin{aligned} \mathcal{M}_r(L) &\rightarrow \mathcal{M}_r(L \otimes A^{\otimes nr}) \\ E &\mapsto E \otimes A^{\otimes n}. \end{aligned}$$

Thus, as before, we may assume that, if necessary, by taking a finite fields extension, $\mathcal{M}_{r,d}(C)$ are defined over \mathbf{F}_q as well. Moreover, we know that all \mathbf{F}_q -rational semi-stable vector bundles are indeed an element in

$$\mathcal{M}_r(C)(\mathbf{F}_q) := \cup_d \mathcal{M}_{r,d}(C)(\mathbf{F}_q).$$

Definition. For $r \in \mathbf{Z}_{>0}$, define a weight r non-abelain zeta function $\zeta_{C,r}(s)$ of C by

$$\zeta_{C,r}(s) := \sum_{E \in \mathcal{M}_r(C)(\mathbf{F}_q)} \frac{q^{h^0(C,E)} - 1}{\#\text{Aut}(E)} \cdot \left(q^{d(E)} \right)^{-s}, \quad \text{Re}(s) > 1.$$

Set also $\xi_{C,r}(s) := q^{r(g-1)s} \cdot \zeta_{C,r}(s)$, $t = q^{-s}$ and $Z_{C,r}(t) = \zeta_{C,r}(s)$.

Theorem 2. (a) $\zeta_{C,r}(s)$ is absolutely convergent for $\text{Re}(s) > 1$, can be meromorphically extended to the whole complex $t = q^{-s}$ plane which has simple poles at $s = 0$ and $s = 1$ with the same residue, and satisfies the functional equation $\xi_{r,L}(s) = \xi_{r,L}(1-s)$.

(b) $Z_{r,C}(t) = \frac{P(t)}{(1-t^r)(1-(qt)^r)}$ where $P(t) \in \mathbf{Q}[t]$ is a degree $2rg$ polynomial with rational coefficients.

(c) Let $P(t) = P(0) \cdot \prod_{i=1}^{2rg} (1 - \omega_i t)$, then after a suitable arrangement, we have

$$\omega_i \cdot \omega_{2rg-i} = q, \quad i = 1, \dots, rg.$$

From the proof given below, one can give the precise values of the residues in terms of a certain combination of Harder-Narasimhan numbers. We leave this to the reader. Moreover, (c) suggests that Weil-Riemann Hypothesis ([W]) holds also for our new zeta functions. This then leads to the following

Riemann-Weil Hypothesis. The reciprocal roots ω_i , $i = 1, \dots, 2rg$, of the weight r non-abelain zeta functions of curves defined over finite fields satisfy

$$|\omega_i| = q^{\frac{1}{2}}, \quad i = 1, \dots, 2rg.$$

Proof of the Theorem. By a finite field extension, we may assume that all $\mathcal{M}_r(L)$ are defined over \mathbf{F}_q if L is defined over \mathbf{F}_q . With this, (a) is a direct consequence of Theorem 1 in §1 by the fact that

$$\mathcal{M}_{r,d}(C)(\mathbf{F}_q) = \cup_{d=0}^{r-1} \cup_{L \in \text{Pic}^d(C)(\mathbf{F}_q)} \cup_{k=1}^{g-1} \mathcal{M}_{r,L \otimes A^{kr}}(\mathbf{F}_q).$$

Hence we only need to prove (b) and (c).

Let us assume (b), then by the functional equation for $t = q^{-s}$, which changes t to $\frac{1}{qt}$, $\prod_{i=1}^{2rg}(1 - \omega_i t)$ changes to $\prod_{i=1}^{2rg}(1 - \frac{\omega_i}{qt})$ accordingly. Thus

$$\prod_{i=1}^{2rg}(1 - \omega_i t) = \prod_{k=1}^{2rg}(1 - \frac{1}{\omega_k q} t).$$

This then implies (c).

So it suffices to prove (b), the rationality of our non-abelian zeta function. For this, let us first assume that

$$Z_{r,C}(t) = \frac{P(t)}{(1-t^r)(1-(qt)^r)}$$

where $P(t) \in \mathbf{Q}[t]$. Then by the functional equation for $\zeta_{r,C}(s)$, we have

$$\frac{P(t)}{(1-t^r)(1-(qt)^r)} \cdot t^{-r(g-1)} = \frac{P(\frac{1}{qt})}{(1-\frac{1}{(qt)^r})(1-\frac{1}{t^r})} \cdot (qt)^{r(g-1)}.$$

Thus, $\deg(P) - 2r - r(g-1) = r(g-1)$ by comparing degrees of t of rational functions on both sides. That is to say, we have $\deg(P) = 2rg$. In this way, we are lead to prove the following

Theorem'. *With the same notation as above, there exists a polynomial $P(t) \in \mathbf{Q}[t]$ such that*

$$Z_{r,C}(t) = \frac{P(t)}{(1-t^r)(1-(qt)^r)}.$$

Proof. Note that we have the following vanishing: for E semi-stable, if $d(E) > r(2g-2)$, then $h^1(C, E) = 0$; while if $d(E) < 0$, then $h^0(C, E) = 0$. Thus, by definition,

$$\begin{aligned} & Z_{r,C}(t) \\ &= \sum_{E \in \mathcal{M}_r(C)(\mathbf{F}_q), d(E) \geq 0} \frac{q^{h^0(C,E)} - 1}{\#\text{Aut}(E)} \cdot (q^{d(E)})^{-s} \\ &= \sum_{E \in \mathcal{M}_r(C)(\mathbf{F}_q), d(E) \geq 0} \frac{q^{h^0(C,E)}}{\#\text{Aut}(E)} \cdot (q^{d(E)})^{-s} - \sum_{E \in \mathcal{M}_r(C)(\mathbf{F}_q), d(E) \geq 0} \frac{1}{\#\text{Aut}(E)} \cdot (q^{d(E)})^{-s} \\ &= \sum_{E \in \mathcal{M}_r(C)(\mathbf{F}_q), r(2g-2) \geq d(E) \geq 0} \frac{q^{h^0(C,E)}}{\#\text{Aut}(E)} \cdot (q^{d(E)})^{-s} + \sum_{E \in \mathcal{M}_r(C)(\mathbf{F}_q), d(E) > r(2g-2)} \frac{q^{h^0(C,E)}}{\#\text{Aut}(E)} \cdot (q^{d(E)})^{-s} \\ &\quad - \sum_{E \in \mathcal{M}_r(C)(\mathbf{F}_q), d(E) \geq 0} \frac{1}{\#\text{Aut}(E)} \cdot (q^{d(E)})^{-s}. \end{aligned}$$

Set

$$\begin{aligned} I &:= \sum_{E \in \mathcal{M}_r(C)(\mathbf{F}_q), r(2g-2) \geq d(E) \geq 0} \frac{q^{h^0(C,E)}}{\#\text{Aut}(E)} \cdot (q^{d(E)})^{-s} \\ II &:= \sum_{E \in \mathcal{M}_r(C)(\mathbf{F}_q), d(E) > r(2g-2)} \frac{q^{h^0(C,E)}}{\#\text{Aut}(E)} \cdot (q^{d(E)})^{-s} \\ III &:= \sum_{E \in \mathcal{M}_r(C)(\mathbf{F}_q), d(E) \geq 0} \frac{1}{\#\text{Aut}(E)} \cdot (q^{d(E)})^{-s} \end{aligned}$$

Then

$$Z_{r,C}(t) = I(t) + II(t) + III(t).$$

Thus it suffices to prove the following

Lemma. *With the same notation as above,*

$$I(t), \quad (1 - (qt)^r) \cdot II(t), \quad \text{and} \quad (1 - t^r) \cdot III(t) \in \mathbf{Q}[t].$$

Proof of the Lemma. By definition, $I(t) \in \mathbf{Q}[t]$. Hence we should prove the assertions for $II(t)$ and $III(t)$.

We first deal with $II(t)$. By the vanishing, we have $h^0(C, E) = d - r(g - 1)$. Hence

$$II(t) = q^{-r(g-1)} \sum_{E \in \mathcal{M}_r(C)(\mathbf{F}_q), d(E) > r(2g-2)} \frac{(qt)^{d(E)}}{\#\text{Aut}(E)}.$$

Let $\mathcal{M}_{r,k} := \mathcal{M}_{r,k}(C)(\mathbf{F}_q)$, then by tensoring with $A^{\otimes n}$, $n \in \mathbf{Z}_{\geq 0}$, we have isomorphisms

$$\begin{aligned} \mathcal{M}_{r,0} &\simeq \mathcal{M}_{r,r} \simeq \dots \simeq \mathcal{M}_{r,r(2g-2)} \simeq \mathcal{M}_{r,(r+1)(2g-2)} \simeq \dots \\ \mathcal{M}_{r,1} &\simeq \mathcal{M}_{r,r+1} \simeq \dots \simeq \mathcal{M}_{r,r(2g-2)+1} \simeq \mathcal{M}_{r,(r+1)(2g-2)+1} \simeq \dots \\ &\dots \quad \dots \quad \dots \\ \mathcal{M}_{r,r-1} &\simeq \mathcal{M}_{r,r+(r-1)} \simeq \dots \simeq \mathcal{M}_{r,r(2g-2)+(r-1)} \simeq \mathcal{M}_{r,(r+1)(2g-2)+(r-1)} \simeq \dots \end{aligned}$$

Thus, if we further set

$$M_r := \mathcal{M}_{r,r(2g-2)+1} \cup \dots \cup \mathcal{M}_{r,(r+1)(2g-2)+(r-1)} \cup \mathcal{M}_{r,(r+1)(2g-2)},$$

then

$$\begin{aligned} II(t) &= q^{-r(g-1)} \sum_{E \in M_r} \sum_{n=0}^{\infty} \frac{(qt)^{d(E \otimes A^{\otimes n})}}{\#\text{Aut}(E)} \\ &= q^{-r(g-1)} \sum_{E \in M_r} \sum_{n=0}^{\infty} \frac{(qt)^{d(E) + nr}}{\#\text{Aut}(E)} = q^{-r(g-1)} \sum_{E \in M_r} \frac{(qt)^{d(E)}}{\#\text{Aut}(E)} \cdot \sum_{n=0}^{\infty} ((qt)^r)^n \\ &= q^{-r(g-1)} \sum_{E \in M_r} \frac{(qt)^{d(E)}}{\#\text{Aut}(E)} \cdot \frac{1}{1 - (qt)^r}. \end{aligned}$$

This proves the assertion for $II(t)$.

Similarly, for $III(t)$, set

$$M'_r := \mathcal{M}_{r,0} \cup \mathcal{M}_{r,1} \cup \dots \cup \mathcal{M}_{r,r-1}.$$

Then

$$\begin{aligned} III(t) &= \sum_{E \in M'_r} \sum_{n=0}^{\infty} \frac{t^{d(E \otimes A^{\otimes n})}}{\#\text{Aut}(E)} \\ &= \sum_{E \in M'_r} \sum_{n=0}^{\infty} \frac{t^{d(E) + nr}}{\#\text{Aut}(E)} = \sum_{E \in M_r} \frac{t^{d(E)}}{\#\text{Aut}(E)} \cdot \sum_{n=0}^{\infty} (t^r)^n \\ &= \sum_{E \in M'_r} \frac{t^{d(E)}}{\#\text{Aut}(E)} \cdot \frac{1}{1 - t^r}. \end{aligned}$$

This proves the assertion for $III(t)$. All in all, we have proved the lemma, hence the theorem' and the theorem itself.

As a direct consequence of the rationality of our new zeta functions, i.e., (b) of the Theorem, we have the following

Corollary. (a) For each $m \geq 1$ there exists suitable number N_m such that

$$Z_{r,C}(t) = \exp \left(\sum_{m=1}^{\infty} N_m \frac{t^m}{m} \right). \quad (**)$$

Moreover,

$$N_m = \begin{cases} r(1 + q^m) - \sum_{i=1}^{2rg} \omega_i^m, & \text{if } r \mid m; \\ - \sum_{i=1}^{2rg} \omega_i^m, & \text{if } r \nmid m. \end{cases}$$

(b) For any positive integer a such that $(a, r) = 1$, if $\zeta_{i,a}, i = 1, \dots, a$ denote all a -th roots of unity, then

$$\prod_{i=1}^a Z_{C,r}(\zeta_{i,a} t) = \exp \left(\sum_{m=1}^{\infty} N_{ma} \frac{T^m}{m} \right) \quad (***)$$

where $T = t^a$.

Proof. In fact, note that $\log(1 - x) = -\sum_{m=1}^{\infty} \frac{x^m}{m}$, from the rationality of $Z_{C,r}(t)$, we get (**) directly. As for the precise formula of N_m , we use the following fact which also implies (b) directly:

$$\sum_{i=1}^a \zeta_{i,a}^m = \begin{cases} a, & \text{if } a \mid m, \\ 0, & \text{if } a \nmid m. \end{cases}$$

Clearly, (***) suggests that

$$\prod_{i=1}^a Z_{C,r}(\zeta_{i,a} t) = Z_{C_{ar},r}(T)$$

with C_{ar} is obtained from C by taking simply extension of constant fields from \mathbf{F}_q to $\mathbf{F}_{q^{ar}}$. (Question: Why ar not simply a ?) As a matter of fact, N_1, \dots, N_r do have Diophantine interpretations. Moreover, there are many important relations in this enumerative aspect of moduli spaces of semi-stable vector bundles. As all are associated to the above Riemann-Weil hypothesis, which we cannot verify now, we leave them to some other occasions.

§3. Non-Stable Contributions

In this section, we justify our new non-abelian zeta functions by looking at non-stable contributions. But for simplicity, we only study the case when $r = 2$.

So let $W_d(C)$ be the collection of isomorphism classes of rank two degree d non-stable vector bundles E (over C) defined over \mathbf{F}_q . Thus clearly, we have then bijections $W_0(C) \simeq W_{2n}(C)$ and $W_1(C) \simeq W_{2n+1}(C)$ for all $n \in \mathbf{Z}$ as for semi-stable vector bundles.

Naturally, one may try to define a new zeta function for C by considering a formal summation

$$\sum_{E \in W_d(C), d \in \mathbf{Z}} \frac{q^{h^0(C,E)} - 1}{\#\text{Aut}(E)} \cdot \left(q^{d(E)} \right)^{-s}, \quad \text{Re}(s) > 1.$$

But this time, we have some troubles. Recall that in semi-stable case, despite that the formal summation is for all $d \in \mathbf{Z}$, finally, only terms with $d \geq 0$ contribute, since $q^{h^0(C,E)} - 1 = 0$ when $d < 0$. Clearly, for non-stable bundles, this does not hold. Hence, we should modify our definition as follows.

Definition. Let C be a regular, geometrically irreducible, projective curve defined over the finite field \mathbf{F}_q with q elements. Define a new zeta functions $\zeta_{C,\text{ns}}(s)$ by setting

$$\zeta_{C,\text{ns}}(s) := \sum_{E \in W_d(C), d \geq 0} \frac{q^{h^0(C,E)} - 1}{\#\text{Aut}(E)} \cdot \left(q^{d(E)} \right)^{-s}, \quad \text{Re}(s) > 1.$$

The main result of this section will be the following

Theorem 3. $\zeta_{C,ns}(s)$ is well-defined and admits a meromorphic extension to the whole complex $t = q^{-s}$ -plane, which is indeed rational too, i.e., can be written as a quotient of two polynomials with rational coefficients.

Proof. Call any element $E \in W_0(C)$ an E_0 and $E_{2n} = E_0 \otimes A^n$ with A as in the previous section. Similarly, call $E \in W_1(C)$ an E_1 and $E_{2n+1} = E_1 \otimes A^n$. Then clearly $\#\text{Aut}(E_*) = \#\text{Aut}(E_{2n+*})$. Moreover, for $E_* \in W_*(C)$, $*$ = 0, 1, by using Harder-Narasimhan filtration, over \mathbf{F}_q , we have the following short exact sequences:

$$0 \rightarrow L_{+,*} \rightarrow E_* \rightarrow L_{-,*} \rightarrow 0 \quad (*)$$

such that

(a_0) if $E_0 \in W_0(C)$, $d_{+,0} := d(L_{+,0}) = 1, 2, 3, \dots$ and $d_{-,0} := d(L_{-,0}) = -d(L_{+,0}) = -1, -2, -3, \dots$;

(a_1) if $E_1 \in W_1(C)$, $d_{+,1} := d(L_{+,1}) = 1, 2, 3, \dots$ and $d_{-,1} := d(L_{-,1}) = 1 - d(L_{+,0}) = 0, -1, -2, \dots$

Thus if we set $L_{+,2n+*} = L_{+,*} \otimes A^n$ with $*$ = 0, 1, then we have also

(a_{2n}) if $E_{2n} \in W_{2n}(C)$, $d_{+,2n} := d(L_{+,2n}) = n+1, n+2, n+3, \dots$ and $d_{-,2n} := d(L_{-,2n}) = n-1, n-2, n-3, \dots$;

(a_{2n+1}) if $E_{2n+1} \in W_{2n+1}(C)$, $d_{+,2n+1} := d(L_{+,2n+1}) = n+1, n+2, n+3, \dots$ and $d_{-,2n+1} := d(L_{-,2n+1}) = n, n-1, n-2, \dots$

Next, we compute $\#\text{Aut}(E)$ according to whether (*) is trivial or not.

(1) If $E_m = L_{+,m} \oplus L_{-,m}$, the automorphisms consist of $\mathbf{F}_q^* \times \mathbf{F}_q^*$ together with the unipotents of the form $1 + \phi$ with

$$\phi \in \text{Hom}(L_{-,m}, L_{+,m}) = H^0(C, L_{-,m}^{\otimes -1} \otimes L_{+,m}) = \begin{cases} H^0(C, L_{-,0}^{\otimes -1} \otimes L_{+,0}), & \text{if } m = 2n; \\ H^0(C, L_{-,1}^{\otimes -1} \otimes L_{+,1}), & \text{if } m = 2n + 1. \end{cases}$$

Hence

$$\#\text{Aut}(E_m) = (q-1)^2 \cdot \begin{cases} q^{h^0(C, L_{-,0}^{\otimes -1} \otimes L_{+,0})}, & \text{if } m = 2n; \\ q^{h^0(C, L_{-,1}^{\otimes -1} \otimes L_{+,1})}, & \text{if } m = 2n + 1. \end{cases}$$

(2) For non-trivial extensions, we have only one copy of \mathbf{F}_q^* and hence

$$\#\text{Aut}(E_m) = (q-1) \cdot \begin{cases} q^{h^0(C, L_{-,0}^{\otimes -1} \otimes L_{+,0})}, & \text{if } m = 2n; \\ q^{h^0(C, L_{-,1}^{\otimes -1} \otimes L_{+,1})}, & \text{if } m = 2n + 1. \end{cases}$$

But non-trivial extensions E_m correspond to non-zero elements of $H^1(C, L_{-,m}^{\otimes -1} \otimes L_{+,m})$ and proportional vectors give isomorphic bundles. Hence the number of isomorphism classes of bundles E_m for which E is non-trivial is

$$\frac{q^{h^1(C, L_{-,m}^{\otimes -1} \otimes L_{+,m})} - 1}{q-1}.$$

Thus in particular, in the summation

$$\sum_{E \in W_d(C), d \geq 0} \frac{1}{\#\text{Aut}(E)} (q^{d(E)})^{-s},$$

the contributions arising from given $L_{+,2n+*}$, $*$ = 0, 1 are

$$\begin{aligned} & \left(\frac{1}{(q-1)^2 q^{h^0(C, L_{-,*}^{\otimes -1} \otimes L_{+,*})}} + \frac{q^{h^1(C, L_{-,*}^{\otimes -1} \otimes L_{+,*})} - 1}{(q-1)^2 q^{h^0(C, L_{-,*}^{\otimes -1} \otimes L_{+,*})}} \right) \cdot (q^{-s})^{2n+*} \\ &= \frac{q^{h^1(C, L_{-,*}^{\otimes -1} \otimes L_{+,*})}}{(q-1)^2 q^{h^0(C, L_{-,*}^{\otimes -1} \otimes L_{+,*})}} \cdot (q^{-s})^{2n+*} \\ &= q^{-d_{+,*} + d_{-,*} + (g-1)} \cdot (q^{-s})^{2n+*} \\ &= \begin{cases} q^{-2d_{+,0} + (g-1)} \cdot (q^{-s})^{2n}, & \text{if } m = 2n; \\ q^{-2d_{+,0} + g} \cdot (q^{-s})^{2n+1}, & \text{if } m = 2n + 1. \end{cases} \end{aligned} \quad (*)$$

(This trick is first used by Atiyah and Bott in [AB] p.595.)

Therefore, if $J_0(C)$ denotes degree zero Jacobian of C , then

$$\begin{aligned}
& \sum_{E \in W_d(C), d \geq 0} \frac{1}{\#\text{Aut}(E)} \left(q^{d(E)} \right)^{-s} \\
&= \#J_0(C)(\mathbf{F}_q) \sum_{d+, * = 1}^{\infty} \sum_{n \geq 0}^{\infty} \cdot \frac{1}{(q-1)^2} \cdot \left(q^{-2d+, 0+(g-1)} \cdot (q^{-s})^{2n} + q^{-2d+, 0+g} \cdot (q^{-s})^{2n+1} \right) \\
&= \#J_0(C)(\mathbf{F}_q) \cdot \frac{1}{(q-1)^2} \cdot \frac{q^{-2}}{1-q^{-2}} \cdot q^{g-1} \left(1 + q^{-2s} + q^{-4s} + \dots + q(q^{-s} + q^{-3s} + q^{-5s} + \dots) \right) \\
&= \frac{q^{g-1}}{(q-1)^2 \cdot (q^2-1)} \cdot \frac{1}{1-q^{-2s}} \cdot (1 + q^{1-s}) \cdot \#J_0(C)(\mathbf{F}_q),
\end{aligned} \tag{**}$$

provided that $\text{Re}(s) > 1$.

Before going further, we reminder the reader that the summation here involves a *double* infinite summations, and hence is quite different in nature comparing with semi-stable cases.

Next, consider the part

$$\sum_{E \in W_d(C), d \geq 0} \frac{q^{h^0(C,E)}}{\#\text{Aut}(E)} \left(q^{d(E)} \right)^{-s}.$$

Similarly as above, we want to group non-trivial extensions and trivial extensions together so that automorphisms may be calculated easily. Clearly for this purpose, then we should assume that $h^0(C, E_m) = h^0(C, L_{+,m}) + h^0(C, L_{-,m})$, which is clearly the case if $h^1(C, L_{+,m}) = 0$, or even better, $d_{+,m} > 2g-2$. By the complete list for degrees at the beginning of this proof, we see that there are only finitely many $E \in W_d(C), d \geq 0$ such that $d_{+,m} \leq 2g-2$. Hence, by definition, up to a polynomial, we are essentially dealing with non-stable bundles such that $h^0(C, E_m) = h^0(C, L_{+,m}) + h^0(C, L_{-,m})$. Moreover, if we can calculate $h^0(C, L_{+,m})$ and $h^0(C, L_{-,m})$ precisely in terms of degrees $d_{+,m}$ and $d_{-,m}$, say in cases when we have vanishing of h^0 or h^1 , proceeding similarly as in (**), we can prove the theorem.

With this in mind, we write

$$\begin{aligned}
& \sum_{E \in W_d(C), d \geq 0} \frac{q^{h^0(C,E)}}{\#\text{Aut}(E)} \left(q^{d(E)} \right)^{-s} \\
&= \sum_{E \in E_* \otimes A^n, E_* \in W_*(C), *, 0, 1, n \leq 2g-1} \frac{q^{h^0(C,E)}}{\#\text{Aut}(E)} \left(q^{d(E)} \right)^{-s} \\
& \quad + \sum_{E \in E_* \otimes A^n, E_* \in W_*(C), *, 0, 1, n \geq 2g} \frac{q^{h^0(C, L_{+, 2n+*}) + h^0(C, L_{-, 2n+*})}}{\#\text{Aut}(E)} \left(q^{d(E)} \right)^{-s} \\
&= \sum_{E \in E_* \otimes A^n, E_* \in W_*(C), *, 0, 1, n \leq 2g-1} \frac{q^{h^0(C,E)}}{\#\text{Aut}(E)} \left(q^{d(E)} \right)^{-s} \\
& \quad + \sum_{E \in E_* \otimes A^n, E_* \in W_*(C), *, 0, 1, n \geq 2g, 0 \leq d_{-,m} \leq 2g-2} \frac{q^{h^0(C, L_{+, 2n+*}) + h^0(C, L_{-, 2n+*})}}{\#\text{Aut}(E)} \left(q^{d(E)} \right)^{-s} \\
& \quad + \sum_{E \in E_* \otimes A^n, E_* \in W_*(C), *, 0, 1, n \geq 2g, d_{-,m} > 2g-2} \frac{q^{\chi(C, E_m)}}{\#\text{Aut}(E)} \left(q^{d(E)} \right)^{-s} \\
& \quad + \sum_{E \in E_* \otimes A^n, E_* \in W_*(C), *, 0, 1, n \geq 2g, d_{-,m} < 0} \frac{q^{\chi(C, L_{+,m})}}{\#\text{Aut}(E)} \left(q^{d(E)} \right)^{-s},
\end{aligned}$$

by a simple observation on cohomology groups. Set now

$$\begin{aligned}
I &:= \sum_{E \in E_* \otimes A^n, E_* \in W_*(C), *, 0, 1, n \geq 2g, d_-, m > 2g-2} \frac{q^{\chi(C, E_m)}}{\# \text{Aut}(E)} \left(q^{d(E)} \right)^{-s}, \\
II &:= \sum_{E \in E_* \otimes A^n, E_* \in W_*(C), *, 0, 1, n \geq 2g, d_-, m < 0} \frac{q^{\chi(C, L_+, m)}}{\# \text{Aut}(E)} \left(q^{d(E)} \right)^{-s}, \\
III &:= \sum_{E \in E_* \otimes A^n, E_* \in W_*(C), *, 0, 1, n \geq 2g, 0 \leq d_-, m \leq 2g-2} \frac{q^{h^0(C, L_+, 2n+*) + h^0(C, L_-, 2n+*)}}{\# \text{Aut}(E)} \left(q^{d(E)} \right)^{-s}, \\
IV &:= \sum_{E \in E_* \otimes A^n, E_* \in W_*(C), *, 0, 1, n \leq 2g-1} \frac{q^{h^0(C, E)}}{\# \text{Aut}(E)} \left(q^{d(E)} \right)^{-s}.
\end{aligned}$$

Then clearly,

$$(i) \quad \sum_{E \in W_d(C), d \geq 0} \frac{q^{h^0(C, E)}}{\# \text{Aut}(E)} \left(q^{d(E)} \right)^{-s} = I + II + III + IV.$$

(ii) By a similar calculation as in (**) for the case $\sum_E \frac{1}{\# \text{Aut}(E)} \left(q^{d(E)} \right)^{-s}$, we see that I and II are convergent provided $\text{Re}(s) > 1$ and are rational.

Therefore, it suffices to deal with III and IV . We study IV first. For this, note that $d_{+, m} + d_{-, m} = m$ or $m + 1$, we have then

$$\begin{aligned}
IV &= \sum_{E \in E_* \otimes A^n, E_* \in W_*(C), *, 0, 1, n \leq 2g-1, d_+, m \leq 2g-2} \frac{q^{h^0(C, E)}}{\# \text{Aut}(E)} \left(q^{d(E)} \right)^{-s} \\
&\quad + \sum_{E \in E_* \otimes A^n, E_* \in W_*(C), *, 0, 1, n \leq 2g-1, d_+, m > 2g-2} \frac{q^{h^0(C, L_+, m) + h^0(C, L_-, m)}}{\# \text{Aut}(E)} \left(q^{d(E)} \right)^{-s} \\
&= \sum_{E \in E_* \otimes A^n, E_* \in W_*(C), *, 0, 1, n \leq 2g-1, d_+, m \leq 2g-2} \frac{q^{h^0(C, E)}}{\# \text{Aut}(E)} \left(q^{d(E)} \right)^{-s} \\
&\quad + \sum_{E \in E_* \otimes A^n, E_* \in W_*(C), *, 0, 1, n \leq 2g-1, d_+, m > 2g-2, 0 \leq d_-, m \leq 2g-1} \frac{q^{h^0(C, L_+, m) + h^0(C, L_-, m)}}{\# \text{Aut}(E)} \left(q^{d(E)} \right)^{-s} \\
&\quad + \sum_{E \in E_* \otimes A^n, E_* \in W_*(C), *, 0, 1, n \leq 2g-1, d_+, m > 2g-2, d_-, m < 0} \frac{q^{\chi(C, L_+, m)}}{\# \text{Aut}(E)} \left(q^{d(E)} \right)^{-s}.
\end{aligned}$$

Now set

$$\begin{aligned}
IV_a &:= \sum_{E \in E_* \otimes A^n, E_* \in W_*(C), *, 0, 1, n \leq 2g-1, d_+, m \leq 2g-2} \frac{q^{h^0(C, E)}}{\# \text{Aut}(E)} \left(q^{d(E)} \right)^{-s} \\
IV_b &:= \sum_{E \in E_* \otimes A^n, E_* \in W_*(C), *, 0, 1, n \leq 2g-1, d_+, m > 2g-2, 0 \leq d_-, m \leq 2g-1} \frac{q^{h^0(C, L_+, m) + h^0(C, L_-, m)}}{\# \text{Aut}(E)} \left(q^{d(E)} \right)^{-s} \\
IV_c &:= \sum_{E \in E_* \otimes A^n, E_* \in W_*(C), *, 0, 1, n \leq 2g-1, d_+, m > 2g-2, d_-, m < 0} \frac{q^{\chi(C, L_+, m)}}{\# \text{Aut}(E)} \left(q^{d(E)} \right)^{-s}.
\end{aligned}$$

Then

- (a) $IV = IV_a + IV_b + IV_c$;
- (b) IV_a and IV_b consist of only finitely many terms and hence are polynomials;

(c) IV_c may be calculated similarly as in the case $\sum_E \frac{1}{\#\text{Aut}(E)} \left(q^{d(E)} \right)^{-s}$.

Hence we get the convergence provided $\text{Re}(s) > 1$ and the rationality for IV .

Finally, let us deal with III . This seems to be a rather delicate work: the summation is an infinite one; while previous trick cannot be applied since we cannot calculate $h^0(C, L_{-,m})$ precisely. However, we are quite lucky: In any case, now we have $h^0(C, E_m) = h^0(C, L_{+,m}) + h^0(C, L_{-,m})$ and $h^0(C, L_{+,m}) = \chi(C, L_{+,m})$. Hence,

$$III = \sum_{E \in E_* \otimes A^n, E_* \in W_*(C), *, n \geq 2g, 0 \leq d_{-,m} \leq 2g-2} \frac{q^{h^0(C, L_{-,m}) + d_{-,m}}}{(q-1)^2} \left(q^{2n+*} \right)^{-s}.$$

Set now for $* = 0, 1$,

$$A_*(m) := \sum_{E \in E_* \otimes A^n, E_* \in W_*(C), n \geq 2g, 0 \leq d_{-,m} \leq 2g-2} \frac{q^{h^0(C, L_{-,m}) + d_{-,m}}}{(q-1)^2}.$$

Then we have

$$A_*(m) = \sum_{L \in \text{Pic}^d(C)(\mathbf{F}_q), 0 \leq d \leq 2g-2} \frac{q^{h^0(C, L) + d(L)}}{(q-1)^2} := A(C)$$

which is independent of $*$ and m . Therefore

$$III = A(C) \times \sum_{*, n \geq 2g} \left(q^{2n+*} \right)^{-s},$$

which is easily seen to be convergent if $\text{Re}(s) > 1$ and rational as well. In this way, we complete the proof of the Theorem 3.

In fact, a functional equation also holds for these new zeta functions. But we will leave it to the reader. Clearly, from (the proof of) this theorem, we may conclude that zeta functions associated to non-stable bundles, and hence the so-called total zeta function associated to all bundles, both semi-stable and non-stable, are interesting but not as canonical as that for semi-stable bundles. Moreover, note that total zeta functions are closely related to automorphic L -functions. Hence, it would be of great interests if one does similar decompositions (as what we have done here) for them.

We end this paper by pointing out that this work is a continuation of our small note [We1], in which a different kind of restricted zeta function is formulated, and that all these works are motivated by our new non-abelian zeta functions for number fields [We2].

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